ON THE EFFECT OF A HIGH-FREQUENCY MAGNETIC FIELD ON THE INSTABILITY OF A PLASMA

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The action of a high-frequency magnetic field on low-frequency instabilities of a plasma is considered. The harmonics of the high-frequency field that appear under these conditions are taken into account. It is shown that their effect on the reduction of the growth rate of the instability is weak. However, in analyzing the oscillation spectrum consideration of these harmonics is necessary, since they have the same growth rate as does the suppressed instability. It is well known that in a plasma situated in a strong magnetic field H_{0Z} unstable oscillations develop, the most dangerous being electrostatic oscillations that propagate almost perpendicular to the magnetic field ($k_\perp \gg k_\parallel$). These oscillations have the form of troughs extended almost along H_{0Z}. If it were possible by some method to create conditions under which particles of one species (either electrons or ions) would have time to traverse the distance between the humps of the troughs in a time considerably shorter than the period $2\pi/\omega$ of the unstable oscillations, then the potential of the instability would be smoothed out (i.e., the instability would be suppressed). This may be achieved by exciting a high-frequency magnetic field $H_1(t) = H_1 \cos \Omega t\, (\Omega \gg \omega)$ in the plasma, which is oriented perpendicular to the constant field H_{0Z}. Then the crossing of the humps by the particles is achieved as a result of motion with thermal velocities along the resultant curved magnetic field.

In order to derive the dispersion equation for quasineutral electrostatic oscillations of a plasma that is inhomogeneous along the x coordinate and situated in a magnetic field

$$\mathbf{H}\{0, H_1 \cos \Omega t, H_0\}, \quad H_1 \ll H_0, \quad \Omega \ll \omega_{H\alpha} = e_\alpha H_0 / m_\alpha c$$

let us make use of the drift kinetic equation for the correction f_1^α to the stationary distribution function $f_0^\alpha(v_\parallel)$. (Here $v_\parallel$ is the velocity along the direction of the resultant magnetic field.) [1]

$$\frac{\partial f_1^\alpha}{\partial t} + v_\parallel (\mathbf{h}\nabla) f_1^\alpha + \frac{c}{H_0}([\mathbf{E}\mathbf{h}]\nabla) f_0^\alpha + \frac{e_\alpha}{m_\alpha}(\mathbf{E}\mathbf{h}) \frac{\partial f_0^\alpha}{\partial v_\parallel} = 0 \tag{1}$$

$$\mathbf{E} = -\nabla\varphi, \qquad \mathbf{h} = \mathbf{e}_z + \mathbf{e}_y \frac{H_1}{H_0} \cos \Omega t$$

Using a Fourier transform in the coordinates, we obtain

$$\frac{\partial f_k^\alpha}{\partial t} + i v_\parallel (\mathbf{k}\mathbf{h}) f_k^\alpha - \frac{ic}{H_0} \varphi_k k_y \frac{\partial f_0^\alpha}{\partial x} - i \frac{e_\alpha}{m_\alpha} \varphi_k (\mathbf{k}\mathbf{h}) \frac{\partial f_0^\alpha}{\partial v_\parallel} = 0 \tag{2}$$

Let us integrate (2) with respect to t by the method of variation of an arbitrary constant

$$f_k^\alpha(v_\parallel, t) = \frac{e_\alpha}{m_\alpha} \frac{1}{v_\parallel} \frac{\partial f_0^\alpha}{\partial v_\parallel} \varphi_k - \frac{e_\alpha}{m_\alpha} \frac{1}{v_\parallel} \frac{\partial f_0^\alpha}{\partial v_\parallel} \int\limits_{-\infty}^{t} \frac{\partial \varphi_k(t')}{\partial t'}$$

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$$\times \exp\left[i\int_{t}^{t'} v_{\|}(\mathbf{kh}(t''))\,dt''\right]dt' + i\frac{ck_y}{H_0}\frac{\partial f_0^{\alpha}}{\partial x}\int_{-\infty}^{t}\varphi_k(t')\exp\left[i\int_{t}^{t'} v_{\|}(\mathbf{kh}(t''))\,dt''\right]dt' \tag{3}$$

We substitute $\varphi_k(t), f_k^{\alpha}(v_{\|}, t)$ into this relationship in the form (compare with [1])

$$\varphi_k(t) = \sum_{n=-\infty}^{\infty}\varphi_n \exp i(-\omega + n\Omega)t,$$

$$f_k^{\alpha}(v_{\|}, t) = \sum_{s=-\infty}^{\infty} f_s^{\alpha}(v_{\|})\exp i(-\omega + s\Omega)t \tag{4}$$

Then, multiplying both sides of the equation by $\exp i(\omega - s\Omega)t$ and integrating over the period $2\pi/\Omega$, we obtain the Fourier component f_s^{α} corresponding to the s-th harmonic of the high-frequency field

$$f_s^{\alpha}(v_{\|}) = \frac{e_{\alpha}}{m_{\alpha}}\frac{1}{v_{\|}}\frac{\partial f_0^{\alpha}}{\partial v_{\|}}\varphi_s + \frac{e_{\alpha}}{m_{\alpha}}\sum_{p,\,l=-\infty}^{+\infty}\varphi_p\left[\frac{1}{v_{\|}}\frac{\partial f_0^{\alpha}}{\partial v_{\|}}(\omega - p\Omega) + \frac{k_y}{\omega_{H\alpha}}\frac{\partial f_0^{\alpha}}{\partial x}\right]\frac{J_l(\mu v_{\|})J_{l+p-s}(\mu v_{\|})}{\Omega(l+p) + k_z v_{\|} - \omega}, \qquad \mu = \frac{k_y H_1}{\Omega H_0} \tag{5}$$

After integrating (5) with respect to $v_{\|}$ we obtain the perturbation of the particle concentration α, which is associated with unstable oscillations:

$$\frac{n_s^{\alpha}}{n_0} = -\frac{e_{\alpha}\varphi_s}{T_{\alpha}} - \frac{e_{\alpha}}{T_{\alpha}}\int_{-\infty}^{+\infty}\sum_{p,\,l=-\infty}^{+\infty}\varphi_p\frac{\omega - p\Omega + \omega_{\alpha}^{*}}{\Omega(p+l) + k_z v_{\|} - \omega}J_l(\mu v_{\|})J_{l+p-s}(\mu v_{\|})f_0^{\alpha}(v_{\|})\,dv_{\|} \tag{6}$$

$$\omega_{\alpha}^{*} = -\frac{ck_y T_{\alpha}}{e_{\alpha}H_0}\frac{\partial \ln n_0}{\partial x}\left(1 - \frac{\eta_{\alpha}}{2} + \eta_{\alpha}\frac{v_{\|}^2}{v_{T\alpha}^2}\right)$$

$$\eta_{\alpha} = \frac{\partial \ln T_{\alpha}}{\partial \ln n_0}, \qquad f_0^{\alpha} = \frac{n_0^{\alpha}(x)}{\sqrt{\pi}v_{T\alpha}}\exp\left(-\frac{m_{\alpha}v_{\|}^2}{2T_{\alpha}(x)}\right)$$

The theoretical aspect of the problem of taking account of the harmonics of the high-frequency magnetic field is clarified using the example of drift-temperature instability ($\omega \ll \omega_{Hi}$, $\omega/k_z \gg v_{Ti}$). In this low-frequency instability the electrons may be considered to be Boltzmann-distributed, i.e., $n_s^e/n_0 = e\varphi_s/T_e$, while the ions may be considered distributed in accordance with Eq. (6). From the condition of quasineutrality of the oscillations ($n_s^e = n_s^i$) we derive an infinite (due to consideration of the harmonics of the high-frequency field) system of equations in φ_s:

$$2\varphi_s + \int_{-\infty}^{\infty}\sum_{p,\,l=-\infty}^{+\infty}\varphi_p\frac{\omega - p\Omega + \omega_i^{*}}{\Omega(l+p) + k_z v_{\|} - \omega}J_l(\mu v_{\|})J_{l+p-s}(\mu v_{\|})f_0^{i}(v_{\|})\,dv_{\|} = 0 \tag{7}$$

$$(T_e \sim T_i)$$

The vanishing of the infinite determinant of this system is what yields the dispersion equation.

Let us prove convergence of the infinite determinant of the system (7). First it is convenient in (7) to calculate the sum with respect to l:

$$\sum_{l=-\infty}^{+\infty}\frac{J_l(\mu v_{\|})J_{l+p-s}(\mu v_{\|})}{\Omega(l+p) + \omega'} = \sum_{l=-\infty}^{+\infty}\frac{J_l J_{l+k}}{l\Omega + \omega''} = -i\sum_{l=-\infty}^{+\infty}\int_0^{\infty}e^{i(\omega''+l\Omega)\tau}J_l J_{l+k}\,d\tau$$

$$= -i(-1)^{k/2}\int_0^{\infty}e^{i(\omega''-k\Omega/2)\tau}J_k(2\mu v_{\|}\sin\Omega\tau/2)\,d\tau$$

Partitioning the infinite integration interval into pieces $(0, \pi)$, $(\pi, 2\pi)$, ... and using the equation [2]

$$\int_0^{\pi} e^{i2\mu x} J_{2\nu}(2\beta\sin x)\,dx = \pi e^{i\mu\pi} J_{\nu-\mu}(\beta)\,J_{\nu+\mu}(\beta), \quad \mathrm{Re}\,\nu > -\frac{1}{2}$$

and the approximation $\omega'/\Omega \ll 1$, we obtain

$$\sum_{l=-\infty}^{\infty} \frac{J_l J_{l+p-s}}{\Omega(l+p)+\omega'} = \frac{1}{\omega'} \begin{cases} (-1)^p J_{p+\omega'/\Omega} J_{-s-\omega'/\Omega} & (p \geqslant s) \\ (-1)^s J_{s+\omega'/\Omega} J_{-p-\omega'/\Omega} & (p \leqslant s) \end{cases}$$

As is well known [3], a determinant converges if: a) the product of its diagonal elements converges absolutely; b) the sum of its nondiagonal elements converges absolutely.

We shall present only the proof of item (a) [item (b) is proved analogously]. For absolute convergence of the product of diagonal elements the absolute convergence of the series

$$\sum_{s=-\infty}^{+\infty} u_s \equiv \sum_{s=-\infty}^{\infty} s J_{s+\omega'/\Omega} J_{-s-\omega'/\Omega}$$

is sufficient.

Let us make use of the equation

$$J_{-s-\omega'/\Omega}(z) = (-1)^s [J_{s+\omega'/\Omega}(z) - \pi\omega'\Omega^{-1} N_{s+\omega'/\Omega}(z)] \qquad (s \geqslant 1)$$

and likewise the asymptotic representation [4] of Bessel functions for $s > z$, which is valid uniformly over all z, $0 < z < \infty$:

$$J_s(z) \approx {}^1/_{12}\pi\sqrt{3\lambda}\, K_{1/3}(sw\lambda), \quad \lambda = {}^1/_3 \ln[(1+w)/(1-w)]$$
$$N_s(z) \approx -\sqrt{\lambda}\,[2I_{1/3}(sw\lambda) + {}^1/_4\pi K_{1/3}(sw\lambda)], \quad w = \sqrt{1 - z^2/s^2}$$

where $K_{1/3}$ $I_{1/3}$ are modified Bessel functions. Then we obtain

$$\lim_{s\to\infty} \frac{|u_{s+1}|}{|u_s|} = 1 - 2s^{-1} + O(s^{-2})$$

which does denote [3] absolute convergence of the series u_s. The series obtained from the one considered by integration with exponential weight will be all the more convergent.

Thus, "cutting up" Bessel functions having the arguments μv_{Ti} by means of an exponential during integration with respect to $v_{\|}$ can easily be seen to ensure rapid convergence of the determinant.

This convergence may be improved if the system φ_s is replaced by another so that the large terms $\sim \Omega/\omega$ vanish from the nondiagonal elements of the determinant of the system (7).

Let us introduce the functions

$$\psi_s^+ = {}^1/_2[\varphi_s + (-1)^s \varphi_{-s}], \quad \psi_s^- = [\varphi_s - (-1)^s \varphi_{-s}]\,\Omega/2\omega \qquad (s = 0, 1, 2, \ldots) \tag{8}$$

that have the properties

$$\psi_{-s}^+ = (-1)^s \psi_s^+, \qquad \psi_{-s}^- = -(-1)^s \psi_s^-$$

Then instead of (7) we have

$$2\psi_s^+ - \sum_{p\geqslant 0} \psi_p^+ \int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_{\|}} J_{-p} J_{-s} f_0^i\, dv_{\|} - \sum_{p>0} \psi_p^+ \int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_{\|}} J_{-p} J_{-s} f_0^i\, dv_{\|}$$

$$+2\sum_{p>0} p\psi_p^- \int_{-\infty}^{\infty} \frac{\omega J_{-p} J_{-s} f_0^i}{\omega - k_z v_\parallel} dv_\parallel + \sum_{\substack{p>0 \\ p+l\neq 0}} \frac{p\psi_p^+}{p+l} \int_{-\infty}^{+\infty} J_l \left[J_{l+p+s} + (-1)^s \times J_{l+p-s}\right] f_0^i dv_\parallel = 0 \quad (9)$$

$$2\psi_s^- - \sum_{\substack{p>0 \\ p+l\neq 0}} \frac{p\psi_p^-}{p+l} \int_{-\infty}^{\infty} J_l \left[J_{l+p-s} - (-1)^s J_{p+l+s}\right] f_0^i dv_\parallel = 0 \quad (10)$$

Here we have taken into account terms of order Ω/ω, 1. Terms of order ω/Ω, however, are discarded.

The system for ψ_s^- is split off. However, if we consider the fact that it does not contain the frequency ω, then ψ_s^- should be assumed equal to zero. The determinant of the remaining system in ψ_s^+, unlike (7), already does not contain larger terms of order Ω/ω and furthermore is semiinfinite.

The convergence of the determinant of the system (9) allows us to limit consideration to a finite number of columns and rows in the determinant in order to obtain the approximate dispersion equation. Let us set the second-order determinant equal to zero:

$$\det \| \Phi_{kl} \| = 0 \qquad (k, l = 0, 1) \quad (11)$$

$$\Phi_{00} = 2 - \int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_\parallel} J_0^2 f_0^i dv_\parallel, \quad \Phi_{10} = \int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_\parallel} J_0 J_1 f_0^i dv_\parallel$$

$$\Phi_{11} = 2 - 2\int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_\parallel} J_1^2 f_0^i dv_\parallel + \sum_{l\neq -1} \frac{1}{l+1} \int_{-\infty}^{+\infty} J_l \left[J_l - J_{l+2}\right] f_0^i dv_\parallel$$

$$\Phi_{01} = 2\int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_\parallel} J_0 J_1 f_0^i dv_\parallel + \sum_{l\neq -1} \frac{2}{l+1} \int_{-\infty}^{\infty} J_l J_{l+1} f_0^i dv_\parallel$$

It is evident that the zero approximation used in [1]

$$\Phi_{00} \equiv 2 - \int_{-\infty}^{\infty} \frac{\omega + \omega_i^*}{\omega - k_z v_\parallel} J_0^2 (\mu v_\parallel) f_0^i (v_\parallel) dv_\parallel = 0 \quad (12)$$

is valid with an accuracy of up to the small term $\Phi_{01}\Phi_{10}$.

This smallness derives from the properties of Bessel functions and the fact that they are cut up into arguments μv_{Ti} by the exponential.

Thus, consideration of the harmonics of the high-frequency field does not lead to qualitatively different results in the sence of influence on the instability growth rate. As was demonstrated in [1], for an amplitude of the high-frequency magnetic field that is not too small (at least for $\mu v_{Ti} = k_y H_1 v_{Ti}/(\Omega H_0 = 2\text{–}3)$] the effect of reduction of the instability growth rate occurs. However, if we speak of the spectrum of the oscillations, then the harmonics of the high-frequency field are substantial. Actually, from the expansion (4) and the vanishing of the determinant of the system (7) it follows that the components φ_n corresponding to the harmonics of the high-frequency field are linearly coupled and build up with the same growth rate. Thus, in the spectrum high-frequencies $\omega + n\Omega$ must be represented besides the low frequency ω, and under these conditions the amplitudes of the spectral lines must fall off with increasing number n. From the point of view of turbulent diffusion the appearance of high frequencies in the spectrum is not dangerous, since diffusion for equality of the growth rates and wave numbers is determined by the low-frequency component of the oscillations [5].

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